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Heat Balance Integral Method for One-Dimensional Finite Ablation

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In this paper, the heat balance integral method is applied to a simple one-dimensional ablation problem. Previous authors have provided solutions that give good approximations over a short time scale; we attempt to provide a solution valid over a large time scale, both before and after ablation begins. We give motivation for the choice of a quartic approximating function $n = 4$ and compare our results with different polynomial approximations and numerical solutions. It is shown that lower values of polynomial degree n provide a better approximation to the temperature profile in the preablation phase, whereas higher values are more accurate during the ablation phase. The temperature gradient at the ablating surface, and consequently the ablation rate, shows a much weaker dependence on n. However, to ensure a positive ablation rate, it is not possible to switch to a higher value of n between preablation and ablation phases. The approximate solutions are compared with numerical and exact analytical solutions whenever possible. Finally, a simple analytical solution is presented that corresponds to the classical solution of Landau.

Nomenclature

c	=	specific heat capacity
E	=	absolute error, $\lvert T_{\text{exact}} - T_{\text{approx}} \rvert$
H	=	thickness of slab
$h(t)$	=	ablation depth
I	=	number of points in spatial grid
k	=	thermal conductivity
L	=	latent heat of ablation
L_2	=	l_2 norm
m	=	exponent of approximating function in ablation stage
N	=	number of points in time grid
n	=	exponent of approximating function in preablation stage
T	=	temperature in ablator
T_A	=	ablation temperature
$T_H(t)$	=	temperature at $x = H$ in ablation stage
T_0	=	initial temperature
t_1	=	end of preablation stage (phase 1)
t_2	=	end of ablation stage (phase 2)
z	=	part of Landau transformation $[=(H-h)^2]$
α	=	thermal diffusivity
Δt	=	time step
$\Delta\xi$	=	transformed spatial step
$\delta(t)$	=	heat penetration depth
δ_1	=	value of δ at $t = t_1$
ξ	=	Landau coordinate
ρ	=	density

Subscript

i	=	spatial index for numerical solution

Superscript

n	=	time index for numerical solution

I. Introduction

ABLATION is the process whereby mass is removed from an object by vaporization or other similar erosive processes. Perhaps the classic example involves heat shields on space vehicles. The exposed surface of the ablative material is designed to burn off, and the resultant gases will carry much of the heat away, whereas the remaining material acts as an insulator. Ablation also occurs in other branches of physics, for example, in the melting or sublimation of a solid or laser drilling in metals and the cornea (see Lin [1] for example).

In this paper, we describe the use of the heat balance integral method (HBIM) to the basic problem of ablating a finite one-dimensional layer [2,3]. Goodman first introduced the HBIM to thermal problems with specific boundary conditions using low-order polynomial approximations. His approach was an adaptation of the Karman–Pohlhausen integral method for analyzing boundary layers (see Schlichting [4]). We focus on the case in which, at the ablating surface, there is a constant input of energy, for example, through aerodynamic heating. The energy sources on an in-flight aircraft surface are discussed in more detail in Myers et al. [5,6]. The other surface is in contact with a perfect insulator. Landau [7] solved this problem numerically and also showed that a quasi-steady solution exists in which the rate of ablation is constant. Goodman [8] applied the HBIM to this problem and obtained the same growth rate as Landau. Zien [9] adapted the method by employing an exponential temperature profile. His results show better agreement with Landau's numerical result for the ablation rate than the standard HBIM. He also presents results when the input energy is a monotonically increasing function of time. Recently, Braga et al. [10,11] have taken a more standard HBIM approach in which the approximating temperature function is again a polynomial but with the order determined by comparing the time ablation commences with standard exact analytical solutions.

Perhaps the greatest deficiency of the HBIM is its dependence on the choice of approximating function (see Bell [12] and Langford [13], for example). Wood [14] showed there are six possible choices even for the basic problem introduced by Goodman [8], and that the choice made by Goodman is in fact only the third best for the majority of problems. Langford [13] investigates a variety of approximating polynomials, from linear to quartic, and proposes measuring the error

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through a least-squares approximation of the original heat equation. Braga et al. [10] use a noninteger order polynomial for the preablation phase in which the order is determined from the exact solution. Of course this then restricts the method to problems in which an exact solution is known. Once ablation occurs, this choice of n proves inadequate, and they switch to a higher value that provides a better match with their numerical solution. However, no criteria is provided for this choice.

The advantage of HBIM is that it significantly reduces the complexity of the problem. The original problem is governed by a heat equation coupled to a Stefan condition that determines the domain over which the heat equation is applied. The HBIM solution involves integrating two first-order differential equations in time. HBIM is therefore a relatively simple technique and is easily adapted to different boundary conditions. Our aim is to develop an HBIM that provides an accurate and consistent description of ablation. In doing so, we show that the approximating polynomial used by Braga and Mantelli gives the best approximation to the temperature when $t \approx t_1$, but this is not the case for later or earlier times. Furthermore, increasing the value of n between pre- and postablation leads to an increase in the ablator thickness for a short time after the switch (i.e., mass is added near $t = t_1$). We, therefore, demonstrate that the best choice of approximating polynomial has the same order before and after ablation commences. We also show that although the choice of n is crucial in obtaining an accurate temperature profile, the temperature gradient near the ablating front is relatively insensitive to n. It is the temperature gradient that drives the ablation. Consequently, there will be a range of n that provides an accurate description of the position of the ablating front. The accuracy of our method is evaluated by comparison with a numerical solution developed by Kutluay et al. [15]. We restrict our analysis in the current paper to the base problem in which the ablation is due to a constant influx of heat at the surface. However, the method can easily be extended to more complex boundary conditions, even for which there is no analytical solution during the preablation phase.

In the following section, we formulate the mathematical description of the problem. We then give the exact analytical solution for the preablation phase. This solution is used to motivate our approximating function in a similar manner to that employed in [16]. In Sec. VI, we compare approximate and numerical solutions and show that the choice of n can significantly affect the temperature profile, but it has a relatively small affect on the temperature gradient at the ablating surface and, consequently, the ablation depth is relatively unaffected by the choice of approximating polynomial. Finally, we present an approximate analytical expression for the position of the ablating surface in terms of the input parameters.

II. Problem Statement and Governing Equations

Consider the one-dimensional ablation problem in which a block of ablative material of thickness H, such as Teflon®, is heated at one surface while the other surface is in contact with an insulator, as shown on Fig. 1. The temperature of the ablator is denoted by $T(x, t)$ and, initially, the temperature is constant $T(x, 0) = T_0$, where T_0 is below the ablation temperature $T_0 < T_A$. We assume that the process

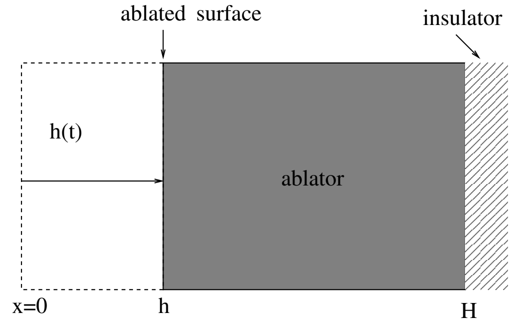

Fig. 1 Problem configuration.

occurs in three distinct phases, as shown on Fig. 2. In the preablation stage (phase 1), the heat penetrates the material and raises its temperature above the initial temperature T_0 in a region of length $\delta(t)$ known as the *heat penetration depth*, where $\delta(0) = 0$. Obviously, because the heat equation has infinite speed of propagation, the distance δ is a fictitious measure denoting the point at which the temperature rise is negligible. This stage ends at time t_1, when the temperature at $x = 0$ reaches the ablating temperature $T(0, t_1) = T_A$. Because the ablator is a poor conductor, we assume that this occurs before $\delta = H$. If this is not the case, the following analysis may be easily modified (see Braga et al. [10]).

The ablation period is made up of two separate stages. In the first, which we call phase 2, $\delta \leq H$ and there exists a region in which the temperature remains at T_0. The ablator occupies the region $h(t) < x < H$, where $h(t)$ is known as the *ablation depth*, and the temperature decreases from T_A at $x = h(t)$ to T_0 at $x = \delta(t)$. This stage ends at time $t = t_2$ when $\delta = H$. For $t > t_2$, which we call phase 3, the temperature is everywhere above T_0. This stage continues until the end of the ablation process when all the material has disappeared, and so $h = H$. The length h satisfies $h(t_1) = 0$. Phase 2 ends at $t = t_2$ when δ reaches the right boundary of the ablator H.

The full problem is described by the heat equation coupled to different boundary conditions in the different stages:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $\alpha = k/(\rho c)$ is the thermal diffusivity, ρ is the density, c is the specific heat at constant pressure, and k is the thermal conductivity. We assume that all the thermal properties remain constant throughout the process.

In the preablation stage shown in Fig. 2a, $T(0, t) < T_A$, $0 < t < t_1$, and the boundary conditions are

$$1)\ \frac{\partial T}{\partial x} = -\frac{q}{k}\bigg|_{x=0}, \qquad 2)\ T = T_0|_{x=\delta(t)}, \qquad 3)\ \frac{\partial T}{\partial x} = 0|_{x=\delta(t)} \tag{2}$$

where q is a constant heat flux. The initial conditions are

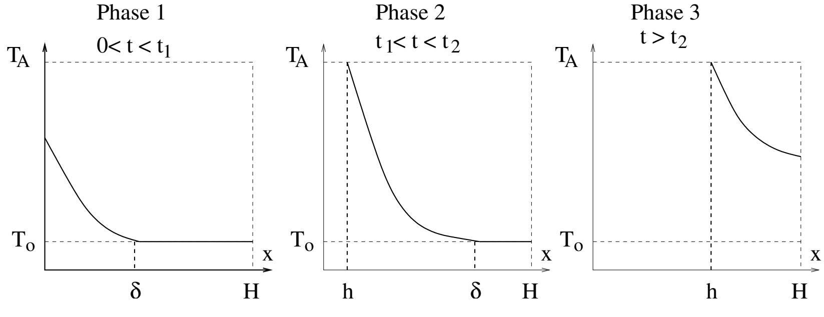

Fig. 2 Schematic of the preablation and ablation stages.

$$1)\ T(x,0) = T_0, \qquad 2)\ \delta(0) = 0 \tag{3}$$

The ablation phase begins at $t = t_1$. The time t_1 is determined by finding the preablation temperature and then solving $T(0, t_1) = T_A$. When $\delta(t) \le H$, the following boundary conditions hold:

$$1)\ T = T_A|_{x=h(t)}, \qquad 2)\ T = T_0|_{x=\delta(t)}, \qquad 3)\ \frac{\partial T}{\partial x} = 0|_{x=\delta(t)} \tag{4}$$

For $\delta \le x \le H$, the temperature is at fixed $T = T_0$. The ablation depth $h(t)$ is determined by the Stefan condition:

$$\rho L \frac{\mathrm{d}h}{\mathrm{d}t} = q + k\frac{\partial T}{\partial x}\bigg|_{x=h(t)} \tag{5}$$

where L is the latent heat of ablation. The initial conditions for the ablation and heat penetration depths are $h(t_1) = 0$ and $\delta(t_1) = \delta_1$, where $\delta_1 > 0$ is the penetration depth at the end of the previous stage. This stage is depicted in Fig. 2b.

Once the heat has penetrated to the end of the ablator, $\delta = H$ and the temperature will increase above T_0 everywhere. We denote the time that this occurs as t_2, and so $\delta(t_2) = H$. The boundary conditions now become

$$1)\ T = T_A|_{x=h(t)}, \qquad 2)\ T = T_H(t)|_{x=H}, \qquad 3)\ \frac{\partial T}{\partial x} = 0|_{x=H} \tag{6}$$

where $T_H(t)$ is an increasing function of time, at present unknown, with $T_H(t_2) = T_0$. This stage is depicted in Fig. 2c.

In the following section, we show how the heat balance integral method may be used to approximate the solution to the previous problem. In the preablation phase, we can determine an analytical solution using Laplace transforms to test the validity of this approximate solution. In the ablation phase, we will again use the HBIM method and compare results with those obtained via a numerical solution of the full problem.

III. Preablation Stage

In the limit $H \to \infty$, we can use Laplace transforms to obtain an exact solution of Eq. (1) with boundary conditions [part 1 of Eq. (2)] and [part 3 of Eq. (2) for $H, \delta(t) \to \infty$] and initial condition [part 1 of Eq. (3)]:

$$T(x,t) = T_0 + \frac{q}{k}\left[2\sqrt{\frac{\alpha t}{\pi}}e^{-x^2/(4\alpha t)} - x\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] \tag{7}$$

The time at which ablation starts is determined by setting $T(0, t_1) = T_A$. Hence,

$$t_1 = \frac{\pi k^2}{4\alpha}\left(\frac{T_A - T_0}{q}\right)^2 \tag{8}$$

Both the temperature and time t_1 can be used to verify the approximate and numerical solutions (provided $\delta < H$). The temperature profile also motivates our choice of approximating function in the heat balance integral method.

If we expand Eq. (7) for small $x/2\sqrt{\alpha t}$, that is, small x or large t, we find

$$T = T_0 + \frac{2q}{k}\sqrt{\frac{\alpha t}{\pi}} - \frac{qx}{k} + \frac{qx^2}{2k\sqrt{\pi\alpha t}} - \frac{qx^4}{48k\sqrt{\pi\alpha^2 t^3}} + \mathcal{O}(x^6) \tag{9}$$

Goodman [8] employs a quadratic approximation when dealing with ablation. We can see that this is in keeping with Eq. (9) to $\mathcal{O}(x^2)$. We will look for a more accurate approximation by retaining terms to $\mathcal{O}(x^4)$. Noting that there is no cubic term in Eq. (9), we, therefore, use the form

$$T = a_1 + b_1(\delta - x) + c_1(\delta - x)^2 + d_1(\delta - x)^4 \tag{10}$$

We write the expression in terms of $\delta - x$ rather than x because this considerably simplifies the algebra. By using a higher order than Goodman, we must introduce a further boundary condition:

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{11}$$

at $x = \delta(t)$. This is consistent with the smooth transition to $T = T_0$ for $x > \delta(t)$. Further discussion of this boundary condition may be found in [14,17]. Applying all boundary conditions, we find

$$T = T_0 + \frac{q}{4k\delta^3}(\delta - x)^4 \tag{12}$$

There is now only the one unknown, $\delta(t)$, to be determined. In Braga et al. [10], a more general form

$$T(x,t) = T_0 + \frac{q}{kn\delta^{n-1}}(\delta - x)^n \tag{13}$$

is employed. To permit comparison with the work of Braga et al. [10], we will adopt this form, and then the solution obtained via Eq. (12) is retrieved by setting $n = 4$.

To determine the thickness of the penetration depth $\delta(t)$, we integrate the heat Eq. (1) from $x = 0$ to $x = \delta$:

$$\alpha\left[\frac{\partial T}{\partial x}\bigg|_{x=\delta} - \frac{\partial T}{\partial x}\bigg|_{x=0}\right] = \int_0^\delta \frac{\partial T}{\partial t}\,\mathrm{d}x = \left[\frac{d}{\mathrm{d}t}\int_0^\delta T\,\mathrm{d}x - T_0\frac{\mathrm{d}\delta}{\mathrm{d}t}\right] \tag{14}$$

Substituting for T from Eq. (13) leads to the following differential equation for δ:

$$\frac{d}{\mathrm{d}t}\left[\frac{\delta^2}{\alpha n(n+1)}\right] = 1 \tag{15}$$

which can be solved using the initial condition $\delta(0) = 0$ to give

$$\delta = \sqrt{\alpha n(n+1)t} \tag{16}$$

Substituting this back into Eq. (13) provides an explicit expression for $T(x,t)$. The approximate time to ablation t_1 is then found by setting $T = T_A$ at $x = 0$:

$$t_1 = \frac{nk^2}{(n+1)\alpha}\left(\frac{T_A - T_0}{q}\right)^2 \tag{17}$$

In Braga et al. [10], the exponent n is chosen so that the expression in Eq. (17) matches the exact expression for t_1 given by Eq. (8), that is

$$n = \frac{\pi}{4 - \pi} \tag{18}$$

This choice of n ensures that the correct ablation time will be predicted by the approximate solution. However, its choice requires the exact solution to be known, in which case t_1 is already known. As we will show later, this choice does not provide the best approximation to the temperature profile, except for when $t \approx t_1$. The temperature profile is important because the ablation height variation depends on T_x, as shown by the Stefan condition in Eq. (5). Our goal in this work is to determine a value of n that gives a good approximation to the temperature profiles as well as t_1 and does not require the use of the exact solution.

IV. Ablation Stage

In this stage, the approximate solution involves two moving boundaries, one at $x = h(t)$ and the fictitious one at $\delta(t)$. The initial conditions are $h(t_1) = 0$ and $\delta(t_1) = \delta_1$, where δ_1 is found using Eq. (16):

$$\delta_1 = \sqrt{\alpha n(n+1)t_1} \tag{19}$$

When $\delta \le H$ the boundary conditions are given by Eqs. (4) and (5). Using the same form for the temperature as in Eq. (13), but with exponent m instead of n (to avoid confusion) and applying the boundary conditions in Eq. (4) gives the following profile for the temperature T:

$$T(x,t) = T_0 + (T_A - T_0)\left(\frac{\delta(t) - x}{\delta(t) - h(t)}\right)^m \tag{20}$$

In an identical manner to the preablation stage, the heat Eq. (1) is integrated from $x = h$ to $x = \delta$. Instead of Eq. (14), we now have

$$\alpha\left[\frac{\partial T}{\partial x}\bigg|_{x=\delta} - \frac{\partial T}{\partial x}\bigg|_{x=h}\right] = \int_h^\delta \frac{\partial T}{\partial t}\,\mathrm{d}x = \frac{d}{\mathrm{d}t}\int_h^\delta T\,\mathrm{d}x - T_0\frac{\mathrm{d}\delta}{\mathrm{d}t} + T_A\frac{\mathrm{d}h}{\mathrm{d}t} \tag{21}$$

Substituting for T from Eq. (20) leads to the following differential equation involving δ and h:

$$m\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\mathrm{d}\delta}{\mathrm{d}t} = \frac{m(m+1)\alpha}{\delta - h} \tag{22}$$

A second equation comes from substituting for $\partial T/\partial x$ in the Stefan condition in Eq. (5):

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{\rho L}\left[q - \frac{km(T_A - T_0)}{\delta - h}\right] \tag{23}$$

The original problem, defined in terms of a partial differential equation (PDE) for the temperature and an ordinary differential equation (ODE) for the interface positions, has now been reduced to two first-order coupled ODEs for h and δ. Analytical and approximate solutions to these equations are discussed in Sec. VII.

Braga et al [10] developed a model where $n = \pi/(4-\pi) \approx 3.66$ in the preablation period, and they subsequently switch to $m = 7$ once ablation commences. No justification is given for this switch, although the reason is most likely that $m = 7$ provides better agreement with the numerical solution of Blackwell and Hogan[18]. Physically, the ablation height must be nonnegative $h \geq 0$. Between $t = 0$ and t_1, the height $h = 0$; subsequently, we require $h_t \geq 0$ to prevent the height becoming negative. From Eq. (23), we see that $h_t \geq 0$ at $t = t_1$, provided

$$q \geq \frac{km(T_A - T_0)}{\delta_1} \tag{24}$$

and, after substitution of t_1 and δ_1 from Eqs. (17) and (19) respectively, this reduces to

$$m \leq n \tag{25}$$

Hence, any switch in the exponent of the approximating function where $m > n$ is invalid and will result in an initial negative growth rate. In the following section, we will show that large values of n are best for large time; smaller values are better for preablation and small $t > t_1$. The best choice for m is, therefore, to set it as high as possible without violating Eq. (25). From now on, we will, therefore, set $m = n$.

The first part of the ablation process (phase 2) ends at time t_2 defined by $\delta(t_2) = H$. For $t > t_2$, the temperature is everywhere greater than T_0. The unknown $\delta(t)$ drops out of the problem and is replaced by the unknown temperature at $x = H$, $T(H,t) = T_H(t)$.

In phase 3, a suitable form for the temperature profile in the region $h < x < H$ is

$$T(x,t) = T_H(t) + [T_A - T_H(t)]\frac{(H-x)^n}{(H-h)^n} \tag{26}$$

To determine h and T_H, we first substitute T into the Stefan condition in Eq. (5) to give an ODE for h:

$$\rho L\frac{\mathrm{d}h}{\mathrm{d}t} = q - \frac{nk(T_A - T_H)}{H - h} \tag{27}$$

The next step is to integrate the heat Eq. (1) from $x = h$ to $x = H$ and substitute for T using Eq. (26). This reduces to

$$-\frac{n(n+1)\alpha(T_A - T_H)}{H - h} = \frac{d}{\mathrm{d}t}[nT_H(H-h) + T_A(H+nh)] \tag{28}$$

The ODEs in Eqs. (27) and (28) must be solved for $t > t_2$ with initial conditions $h(t_2) = h_2$, which is determined as part of the solution in the previous stage, and $T_H(t_2) = T_0$. This stage continues until all the material has ablated with $T(H,t) = T_A$ and $h = H$.

V. Numerical Scheme for Ablation

In the ablation stage, we can find an accurate numerical solution in the following manner. First, we introduce the Landau transformation

$$\xi = \frac{H - x}{H - h} \tag{29}$$

which converts the moving boundary $h(t) < x < H$ to the fixed boundary $0 < \xi < 1$. For obvious reasons, this is known as *boundary immobilization technique* (BIM) and has been shown to give accurate results for Stefan problems, provided the time step is sufficiently small (see Blackwell and Hogan [18], for example).

Under the Landau transformation, the governing equations reduce to

$$\frac{\partial T}{\partial t} = \frac{\xi}{2z}\frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial T}{\partial \xi} + \frac{\alpha}{z}\frac{\partial^2 T}{\partial \xi^2}, \qquad 0 < \xi < 1, \qquad t_1 < t < t_2 \tag{30}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2}{\rho L}\left[-q\sqrt{z} + k\frac{\partial T}{\partial \xi}\right]\bigg|_{\xi=1} \tag{31}$$

where, following Kutluay et al. [15], we replace h with the variable $z = (H-h)^2$. The boundary conditions on Eq. (30) are

$$T = T_A|_{\xi=1}, \qquad \frac{\partial T}{\partial \xi} = 0|_{\xi=0} \tag{32}$$

Initially, $z = H^2$, and the temperature is determined during the preablation calculation.

The mesh and discretized variables are defined by

$$\xi_i = i\triangle\xi, \qquad t^n = t_1 + n\triangle t, \qquad T_i^n = T(\xi_i, t^n), \qquad z^n = z(t^n) \tag{33}$$

where $n = 0, 1, \ldots, N$, and $i = 0, 1, \ldots, I$. The explicit finite difference scheme applied in [15] to Eq. (30) is

$$T_i^{n+1} = T_i^n + \frac{\xi_i \dot{z}^n}{4z^n}\frac{\Delta t}{\Delta \xi}(T_{i+1}^n - T_{i-1}^n) + \frac{\alpha}{z^n}\frac{\Delta t}{\Delta \xi^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) \tag{34}$$

for $n = 0, 1, \ldots, N-1$ and $i = 1, \ldots, I-1$. The boundary condition in the first part of Eq. (32) becomes $T_I^n = T_A$ for $n = 0, 1, \ldots, N$, and the second part of Eq. (32) can be written using central differences as $T_{-1}^n = T_1^n$ for $n = 0, 1, \ldots, N$. This is then substituted into Eq. (34) at $i = 0$ to eliminate the fictitious value T_{-1}^n

$$T_0^{n+1} = T_1^n + \frac{2\alpha}{z^n}\frac{\Delta t}{\Delta \xi^2}\left(T_1^n - T_0^n\right), \qquad n = 0, 1, \ldots, N-1 \tag{35}$$

Finally, the transformed Stefan condition, Eq. (31), is now

$$z^{n+1} = z^n - \frac{2q\Delta t}{\rho L}\sqrt{z^n} + \frac{k\Delta t}{\rho L \Delta \xi}\left(3T_I^n - 4T_{I-1}^n + T_{I-2}^n\right) \qquad n = 0, 1, \ldots, N-1 \tag{36}$$

This three-term backward difference was found by Furzeland [19] to be a suitable replacement for the temperature gradient at a moving interface. The initial conditions are $T_i^0 = T(\xi, t_1)$ and $z^0 = [H - h(t_1)]^2 = H^2$.

Table 1 Physical parameter values for Teflon

Parameter	Typical value	Units
c	1256	J/kg K
k	0.22	W/mK
ρ	1922	kg/m^3
α	9.11×10^{-8}	m^2/s
L	2.326×10^6	J/kg
T_A	833	K
T_0	273	K

VI. Results

We now illustrate the results obtained via the HBIM and compare these with the exact and numerical solutions in the preablation and ablation stages. The parameter values used in all calculations are given in Table 1. Because we only have solutions available for comparison up to t_2, we end the calculation there. Note that for all the results shown in this section, the mesh values for the numerical solution are $N = 500$ and $I = 100$. A mesh refinement shows that the solution converges, and using larger values of N and I does not give a noticeable difference in the solution.

In Fig. 3, we show the absolute percentage error of the temperature profiles at $t = t_1$ for the HBIM solutions (with $n = \pi/(4-\pi)$, 4, 7) compared with the exact solution, Eq. (7). The value of δ increases with n, as shown by Eq. (19). We include the results for $n = 7$ because this is the value used in the ablation stage in Braga et al. [10]. The exact solution predicts $t_1 \approx 0.0327$; with $n = 4$, we find $t_1 \approx 0.0333$ (an error of 1.8%), and when $n = 7$, $t_1 \approx 0.0364$, and the error is around 11%. Near $x = 0$, the curves are all very similar; however, it is clear that $n = 7$ provides the worst agreement. If we define $E = |T_{\text{exact}} - T_{\text{approx}}|$ and calculate the L_2 norm

$$L_2 = \|E\|_2 = \left(\sum E^2\right)^{1/2} \tag{37}$$

then we find $L_2 = 39.6$, 55.8, 208.5 for $n = \pi/(4-\pi)$, 4, 7, respectively. So, $n = \pi/(4-\pi)$ provides the best approximation to the temperature at $t = t_1 \approx 0.03$. However, at $t = 0.02$, $n = 4$ has the lowest value of L_2 and, as we shall see, in the ablation stage, $n = \pi/(4-\pi)$ provides the worst approximation of the three for all time.

In Figs. 4–6, we compare temperature profiles in the ablation stage at $t = t_1$, 0.2, 1, 2, 3, 4 for $n = \pi/(4-\pi)$, 4, 7 with the numerical solution. The position of δ for the three HBIM solutions is marked by an *. At early times (greater than t_1), the smaller values of n give the best approximation. By $t = 1$, $n = 7$ provides excellent agreement with the numerical solution, but after this, the error increases. So, the optimum value of n appears to increase with t. However, it is worth reiterating that to prevent an initial negative growth rate, the exponent of the approximate solution for ablation must be less than or equal to that in the preablation stage. If we choose $n = 7$ for preablation, then the errors in both t_1 and the general temperature profile are large. It is, therefore, undesirable to use $n = 7$ for all time. Further, our numerical experiments indicate that even higher-order

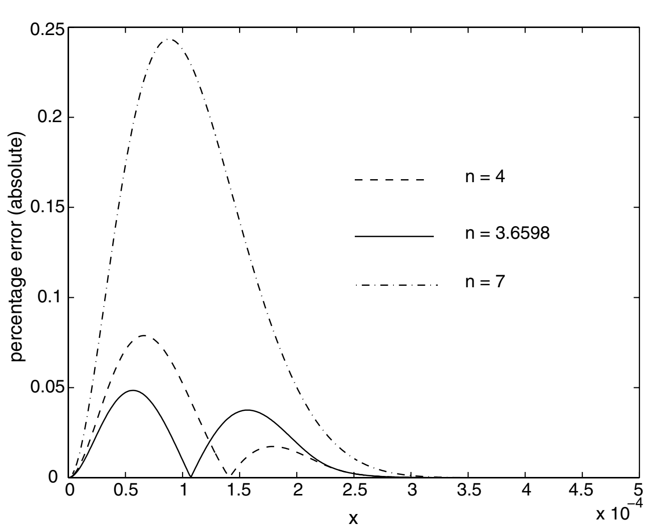

Fig. 3 Percentage error of the temperature at $t = t_1$ for the exact solution, Eq. (7), and the HBIM solutions with $n = \pi/(4-\pi)$, 4, 7.

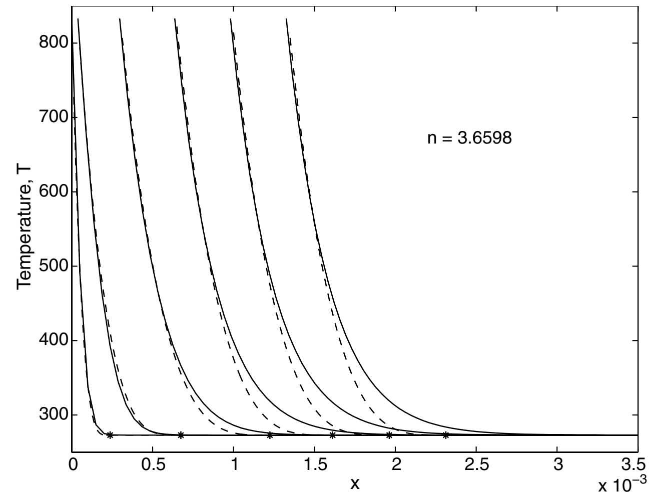

Fig. 4 Phase 2 at $t = t_1$, 0.2, 1, 2, 3, 4, in which the solid line indicates the numerical solution, and the dashed line indicates the HBIM solution.

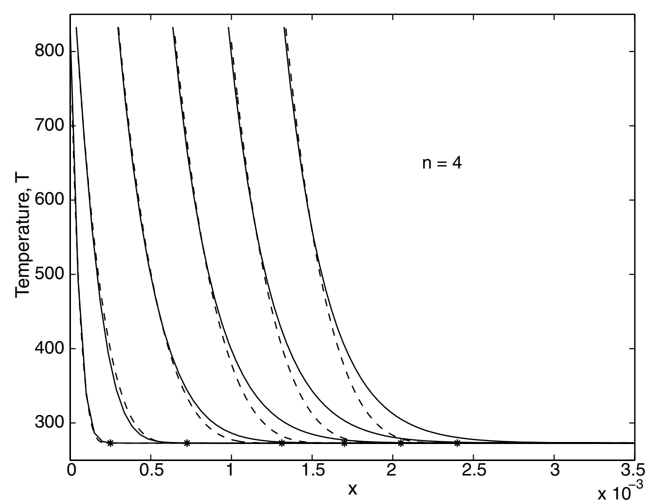

Fig. 5 Phase 2 at $t = t_1$, 0.2, 1, 2, 3, 4, in which the solid line indicates the numerical solution, and the dashed line indicates the HBIM solution.

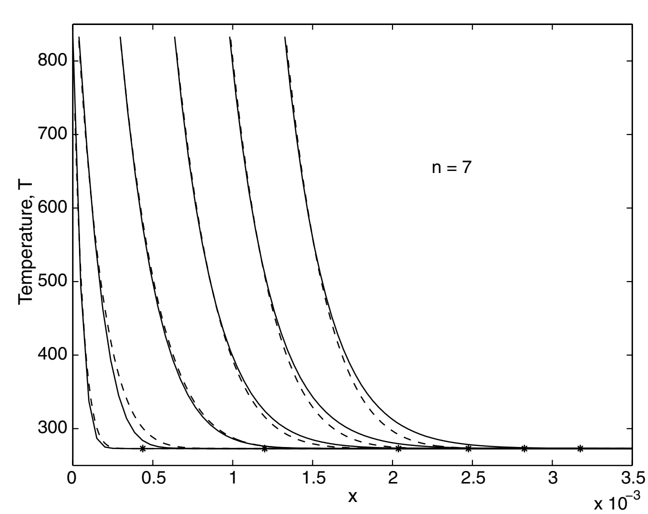

Fig. 6 Phase 2 at $t = t_1$, 0.2, 1, 2, 3, 4, in which the solid line indicates the numerical solution, and the dashed line indicates the HBIM solution.

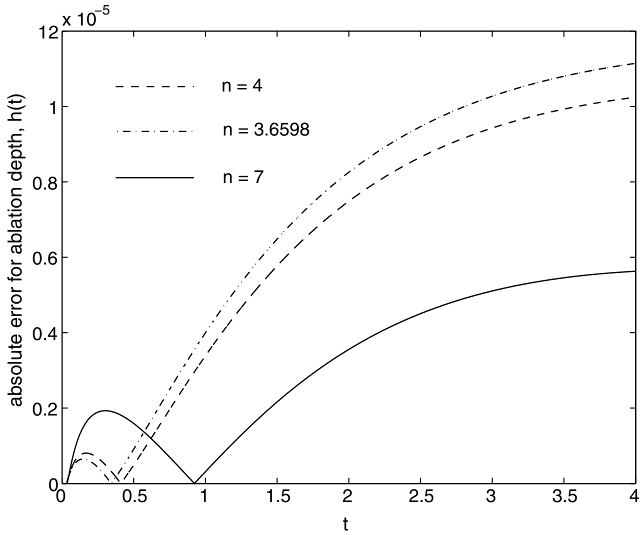

Fig. 7 Absolute error of the height variation during the ablation phase for $n = \pi/(4-\pi)$, 4, 7, compared with the numerical solution.

approximating functions provide more accurate temperature profiles as t increases above 1: $n = 7$ is simply the best choice around $t = 1$.

There are clearly arguments in favor of each of the values of n employed so far (and obviously other values), and it appears that we cannot choose a single value of n to accurately model the temperature profile throughout the process. However, we should bear in mind that it is the depth of ablated material that is of primary interest, and this depends on the accuracy of the temperature gradient at $x = h(t)$. All the profiles appear to give reasonable agreement with the numerical solution near $x = h$. In Fig. 7, we show the variation of h for $t \in [t_1, 4]$. The absolute error is plotted for the three values of n compared with the numerical solution. For very small times $t \in [t_1, 0.48]$, $n = \pi/(4-\pi)$ provides the best approximation. For $t \in [0.48, 0.7]$, $n = 4$ gives the best approximation; subsequently, $n = 7$ has the smallest error. However, even with the worst example at $t = 4$, namely $n = \pi/(4-\pi)$, the error is only around 0.8%. So all values of n employed provide a relatively accurate prediction of the ablation depth.

VII. Simple Approximate Solution for Ablation Stage

In the ablation stage, the problem was reduced to solving Eqs. (22) and (23) (with m now set to n). The solutions may be determined analytically by subtracting $n+1$ times Eq. (23) from Eq. (22) to obtain a separable equation for $y = \delta - h$:

$$\frac{\partial y}{\partial t} = \frac{a}{y} - b, \qquad a = n(n+1)\left[\alpha - \frac{k(T_A - T_0)}{\rho L}\right] \qquad b = \frac{(n+1)q}{\rho L} \tag{38}$$

An implicit solution can be written down for this equation, and the inverse, $\delta - h = f(t)$, leads to the Lambert W function quoted in [10]. However, in general, the Lambert W function has to be calculated numerically and, given the numerical packages currently available, it is probably just as easy to solve Eq. (38) numerically.

A much simpler approximate solution may be found by first writing Eq. (38) in nondimensional form. We denote

$$y = \mathcal{H}y', \qquad h = \mathcal{H}h', \qquad t = \tau t' \tag{39}$$

where $\mathcal{H}$ and τ are unknown height and time scales. The time scale for ablation τ comes from the Stefan condition:

$$\frac{\partial h'}{\partial t'} = \frac{\tau q}{\rho L \mathcal{H}}\left[1 - \frac{kn(T_A - T_0)}{q\mathcal{H}y'}\right] \tag{40}$$

Because it is q that drives the ablation, we see that

$$\tau = \frac{\rho L \mathcal{H}}{q} \tag{41}$$

We take the height scale as the steady-state solution for y, and so $\mathcal{H} = a/b$. Substituting for $\mathcal{H}$, τ, and b in the nondimensional form of Eq. (38), we find

$$\frac{1}{n+1}\frac{\partial y'}{\partial t'} = \frac{1}{y'} - 1 \tag{42}$$

The time derivative term in Eq. (42) is small, and so, to leading order in $1/(n+1)$, the solution is

$$y' \approx 1, \qquad y \approx \frac{a}{b} \tag{43}$$

This means that the width of the heated region is approximately constant. Substituting for y in the Stefan condition in Eq. (23) gives an expression for h; consequently, we can find $\delta = y + h$

$$h = \frac{1}{\rho L}\left[q + \frac{4k(T_A - T_0)b}{a}\right] \qquad t = \frac{qt}{\rho[L - c(T_A - T_0)]} \qquad \delta = \frac{a}{b} + h \tag{44}$$

This is identical to the quasi-steady solution obtained by Landau by setting $T_t = 0$ in the heat equation (see [1,7]).

The numerical solutions to Eqs. (22) and (23) with $n = 4$ are compared with the approximate solutions of Eq. (44) in Fig. 8a. The approximate solutions are shown as dashed lines. The neglect of the time derivative in the y equation means that we cannot satisfy the initial condition in y. This is reflected in the error in δ for small time. However, because δ is a fictitious quantity, this is not of great concern. The important variable is h, which satisfies the correct initial condition and shows an error of less than 2% for all time. The absolute error in δ and h is shown in Fig. 8b. Note that the expression for h is independent of n, and so the error is also independent of n. The expression for δ involves $a/b \sim n$, and so this error does depend on n. Because the neglected term in the nondimensional Eq. (42) is $\mathcal{O}(1/(n+1))$, the error is likely to be greater the smaller the value of n.

VIII. Conclusions

The HBIM has been applied by various authors to the problem of ablation. The choice of approximating function is key to a successful analysis. The most popular choice is a polynomial of order n, and then the question is what value of n provides the most accurate results. Initially, Goodman et al. chose low values, such as $n = 2$ (see Goodman and Shea [3,8,20]). Recently, the method has been refined by Braga et al. [10,21] by choosing n to make the ablation times match with the exact solution. However, this latter method has two main drawbacks. First, it can only be applied to problems in which the exact solution is known. Second, it only ensures agreement with the ablation time. It does not guarantee good agreement with the temperature profile, and, more importantly, the temperature gradient that drives the ablation, for all time. To overcome the first problem, in which an appropriate n cannot be found analytically, in Braga et al. [21], the authors use an average value of n taken from two similar problems with known exact solutions. For the second, they increase n from $\pi/(4-\pi)$ to 7 between the preablation and ablation periods [10].

By comparing our approximate and numerical solutions, we have shown that choosing the same value of n in both the ablation and preablation stages, which does not rely on the exact solution, can provide a better approximation to the temperature profile over a greater time. Importantly, the ablation height, which is the quantity of primary interest, is relatively insensitive to the choice of n. Although, as time increases, a larger value of n will provide a better approximation to the temperature profile throughout the heated region, it is the temperature gradient at $x = h$ that drives the ablation, and this appears insensitive to n. Our numerical experiments indicate that $n = 4$ provides better agreement with the temperature profile over a range of times than the choice motivated by matching with the

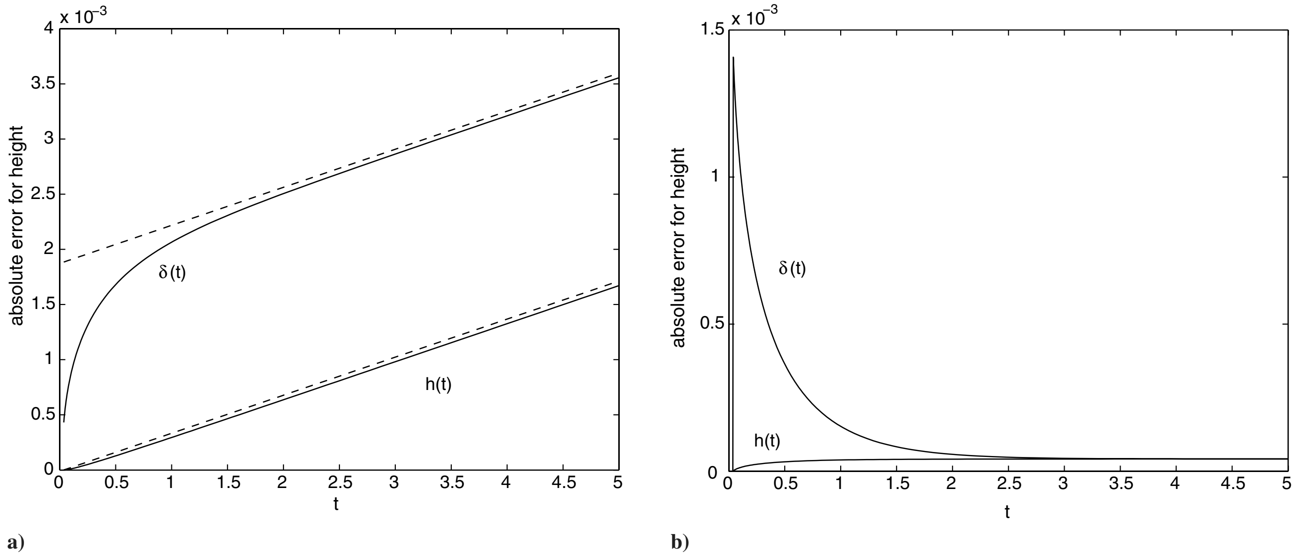

Fig. 8 a) Approximate (dashed line) and exact (solid line) solutions for δ and h. b) Absolute error.

exact solution for $t = t_1$. However, there is clearly a range of n that will provide a reasonably accurate description for $h(t)$.

There is an opportunity to change the value of n when ablation starts because the approximating function changes at this time. Our analysis shows that the growth rate will be positive only if n in the ablation stage is less than or equal to the value in the preablation stage. There is, therefore, a playoff between the high value of n, which gives the more accurate temperature profile for large times but poor prediction of the ablation time, and the small value of n, which appears to work best in preablation. For this reason, we focussed on $n = 4$, which provides satisfactory results in both stages. It can also be used for problems with more complex boundary conditions without relying on an exact solution.

The simple solution, provided in Sec. VII, obtained by nondimensionalizing the governing HBIM equations and solving the leading-order problem, recovered the steady-state solution first presented by Landau. This provides a clear relation between the input parameters and ablation depth.

It is clear that the HBIM can provide excellent and accurate results for ablation problems. However, there will always be the concern about the choice of approximating function, as discussed in [12–14]. We motivated our choice through an exact solution. Although this is not really necessary, it does allow us to gain insight into a possible reason for the lack of accuracy of the temperature profiles as time increases. Our function was chosen by expanding the exact solution for small $\zeta = x/2\sqrt{\alpha t}$. As time increases, the length of the heated region grows in such a way that ζ increases. As $\zeta \to 1$, we will need more terms in the expansion to provide an accurate approximation, and, consequently, we need to increase n. Preliminary investigations have indicated that the correspondence between the approximate and numerical solutions does indeed break down in the vicinity of $\zeta = 1$. In the future, we will focus our attention on approximating functions obtained by expanding the exact solution around an intermediate variable $z = \zeta - 1$.

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